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BIPOLAR HARMONICS ON A CIRCULAR DRUM

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Abstract—A circular drumhead is set in vibration by an impulse at a point not at the center. Instead of solving for the harmonics as a boundary value problem in bipolar coordinates, inversion theory is used to transform it to the circular symmetrical problem which is then solved in terms of cylindrical coordinates. The circles of Apollonius, coaxial with the impulse point, correspond to the concentric circles of the transformed drumhead.

1. INTRODUCTION

When a tympany or kettledrum is played, the impulse point is rarely at the geometrical center. The conductor seems to be able to recognize the difference; hence there must be some noticeable change in the associated harmonics. This paper considers some of these changes. The author's interest in the subject began on viewing the film of Marc Kac lecturing on the topic of hearing the shape of a drum [1]. In this film and the related paper [2], Kac presents arguments indicating that the area and perimeter can possibly be determined through a knowledge of the harmonics. He did not give serious consideration to the effect of the initial conditions, however.

2. THE BOUNDARY VALUE PROBLEM

The problem to be considered is that of determining the nodal lines on a circular drumhead which has been set into vibration by an impulse at a point. The shape of the enclosure will not be considered. The motion is described by the wave equation:

$$\nabla \cdot \nabla f(R,t) = \frac{1}{C^2}\frac{\partial^2 f(R,t)}{\partial t^2}, \tag{1}$$

where f depends on the spatial coordinates, designated by the position vector R, and the time, t. The Laplacian, $\nabla \cdot \nabla$, depends only on the spatial coordinates and has a functional form that depends on the coordinate system employed. On the boundary B, $f(R,t)$ vanishes identically. The initial conditions depend on the method by which the membrane is set into motion. These can be described by the pair of equations

$$f(R,0) = g(R) \qquad \frac{\partial f(R,0)}{\partial t} = h(R). \tag{2}$$

The preferred choice of spatial coordinates is dependent on the shape of the membrane boundary and the initial conditions. Having a circular boundary suggests that the coordinates should have circles as one of the level curves. With such a choice, the boundary conditions can be readily satisfied. If the initial conditions were symmetrical with respect to the center of the drumhead, polar coordinates would be suitable and solutions for this situation can be obtained by separating the variables in the wave equation. The initial conditions of this problem are not symmetric and polar coordinates are not suitable, fortunately another orthogonal coordinate system based on the circle is available, the bipolar coordinate system.

3. BIPOLAR COORDINATES

One of the level curves of the bipolar coordinate system is represented by circles of Apollonius, the other by their orthogonal trajectories which are also circles. The Apollonian circles are described as the locus of points such that the *ratio* of the distance from two fixed points is a constant [3,4]. If the fixed points have Cartesian coordinates $(\pm k,0)$, the equation for the locus of points (x,y) on the circle described by the parameter can be written as

$$x^2 + y^2 + k^2 = 2kx \coth v. \tag{3}$$

These circles have centers at $(k \coth v,\ 0)$ and radius $(k \operatorname{csch} v)$. The coaxial center, $(k,\ 0)$, corresponds to the circle with parameter value $v = \infty$.

The orthogonal trajectories are found by solving the vector differential equation [5]:

$$\mathrm{d}R \cdot \nabla v = 0, \tag{4}$$

where ∇v represents the gradient of the circles of Apollonius. The desired solutions are

$$x^2 + y^2 - k^2 = 2ky \cot u. \tag{5}$$

These circles have centers at $(0,\ k \cot u)$, radius $(k \csc u)$, and pass through the fixed points $(\pm k,\ 0)$ used to define the Apollonian circles. It can be shown that the parameter u corresponds to the angle of inclination of the tangent line to these circles as they pass through the fixed point $(k,\ 0)$.

The wave equation in bipolar coordinates is

$$\frac{(\cosh v - \cos u)^2}{k^2}\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right] = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}. \tag{6}$$

Unfortunately this equation also does not admit to an easy separation of the spatial variables. However, the problem can be transformed to the symmetrical situation through an inversion in an appropriate circle. The variables are then separable.

4. GEOMETRIC INVERSION

The topic of inversion as a transformation has its origins scattered throughout the history of mathematics. Some references to this history can be found in Coxeter [6] and Eves [7].

The inverse of a point P with respect to a circle with center 0 and radius r is defined as the point P' on the line $\overline{OP}$ such that $(\overline{OP})(\overline{OP'}) = r^2$. This transformation is both involutoric and isogonal. If lines are considered as circles of infinite radii, it can be shown that circles invert into circles.

If the boundary of the drumhead is taken as one of the Apollonian circles, these circles can be transformed into a set of concentric polar circles by taking the external fixed point as the center of inversion. This inversion is shown in Fig. 1. The coordinate circles orthogonal to the boundary and passing through the fixed points transform into radial

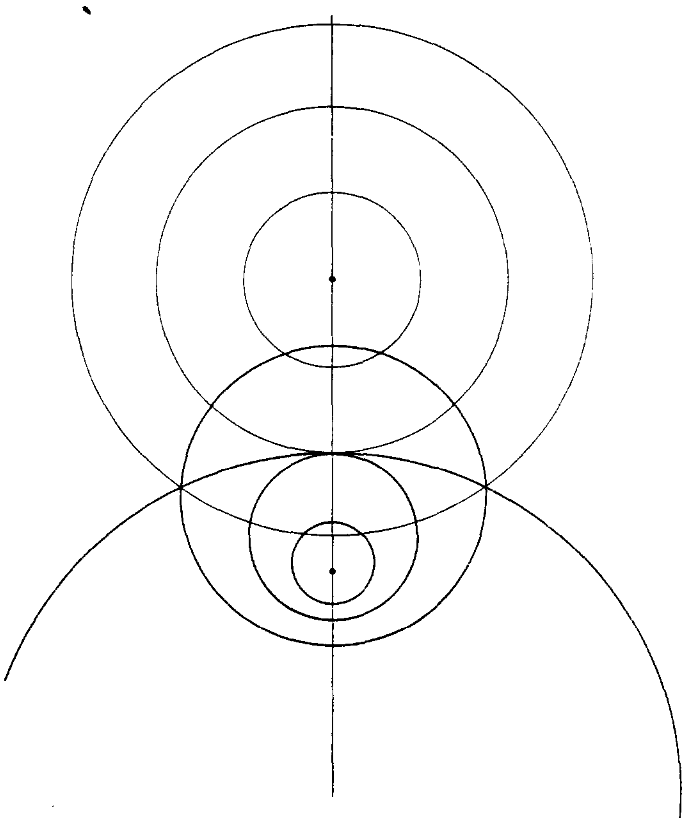

Fig. 1.

lines corresponding to the lines θ = constant in polar coordinates. The equivalence of these coordinates is illustrated in Fig. 2.

The involutoric property of the inversion now makes it possible to solve the problem in polar coordinates and transform the solution back to the appropriate bipolar configuration. The location of the impulse point uniquely determines the parameter value of ν describing the Apollonian circle corresponding to the boundary of the circular membrane.

The relationship of these circles can be described either geometrically or analytically. Analytically, it follows that the family of concentric polar circles with center $(k, 0)$ have the equation

$$(x - k)^2 + y^2 = a^2. \tag{7}$$

Under an inversion in a circle of radius r with center at the origin, these circles map into the Apollonian circles:

$$\left[x - \frac{r^2(1 + \coth\nu)}{2k}\right]^2 + y^2 = \left[\frac{r^2 \operatorname{csch}\nu}{2k}\right]^2, \tag{8}$$

which also have the equation

$$\left[x - \frac{kr^2}{k^2 - a^2}\right]^2 + y^2 = \left[\frac{ar^2}{k^2 - a^2}\right]^2. \tag{9}$$

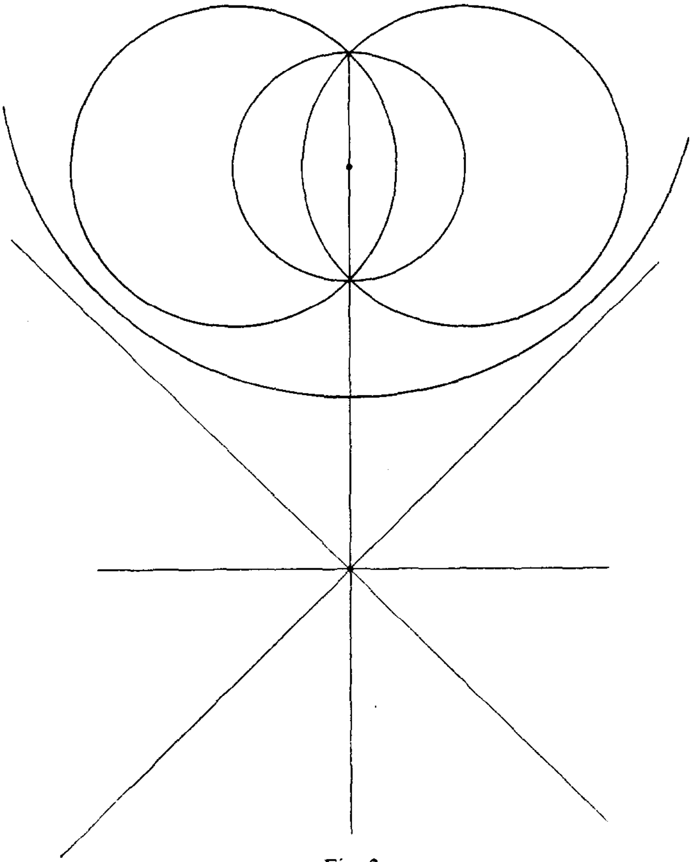

Fig. 2.

From Eqs. (8) and (9) it can be shown that

$$\sinh v = \frac{k^2 - a^2}{2ak} \qquad \cosh v = \frac{k^2 + a^2}{2ak} \tag{10}$$

giving a relationship between parameters a, k, and v. Under the inversion, the center of the concentric circles map into the point $(r^2/k,\ 0)$ which is the coaxial center of the Apollonian circles and the impact point.

Definition.

$$\zeta = \frac{\text{Distance (coaxial center, nearest point to boundary)}}{\text{Radius of Circle}}$$

$$= 1 - \cosh v + \sinh v = (k - a)/k. \tag{11}$$

Using the last form of Eq. (11) the value of a can be determined from the choice of ζ and the parameter k which is arbitrary and may be used to normalize a. Once a is determined the value of v defining the particular Apollonian circle can be calculated.

The choice of r, the radius of the circle of inversion, is also arbitrary. This radius enters into the ratio of the homothety required to transform any given circle of the concentric polar family into the corresponding coaxial bipolar circle. The ratio of the homothety is given by

$$\frac{r^2(1 + \coth \nu)}{2k^2} = \frac{r^2}{k^2 - a^2} = \frac{r^2}{k^2\zeta(2 - \zeta)}. \tag{12}$$

It is apparent that k could also be conveniently used to normalize r.

5. CIRCULAR DRUMHEAD

In solving the proposed problem, the nonsymmetric problem described in terms of bipolar coordinates is transformed to symmetric polar coordinates providing a problem which can be solved in terms of Bessel functions [8]. A plot of the nodal paths for the symmetrical problem is exhibited in Fig. 3. Inverting the nodal paths in the circle of inversion produces the corresponding nodal paths illustrated in Fig. 4. An homothety transformation was carried out so that the diameters of the boundary circles would be the same for both figures. In developing Fig. 3, the impulse point was taken as half way

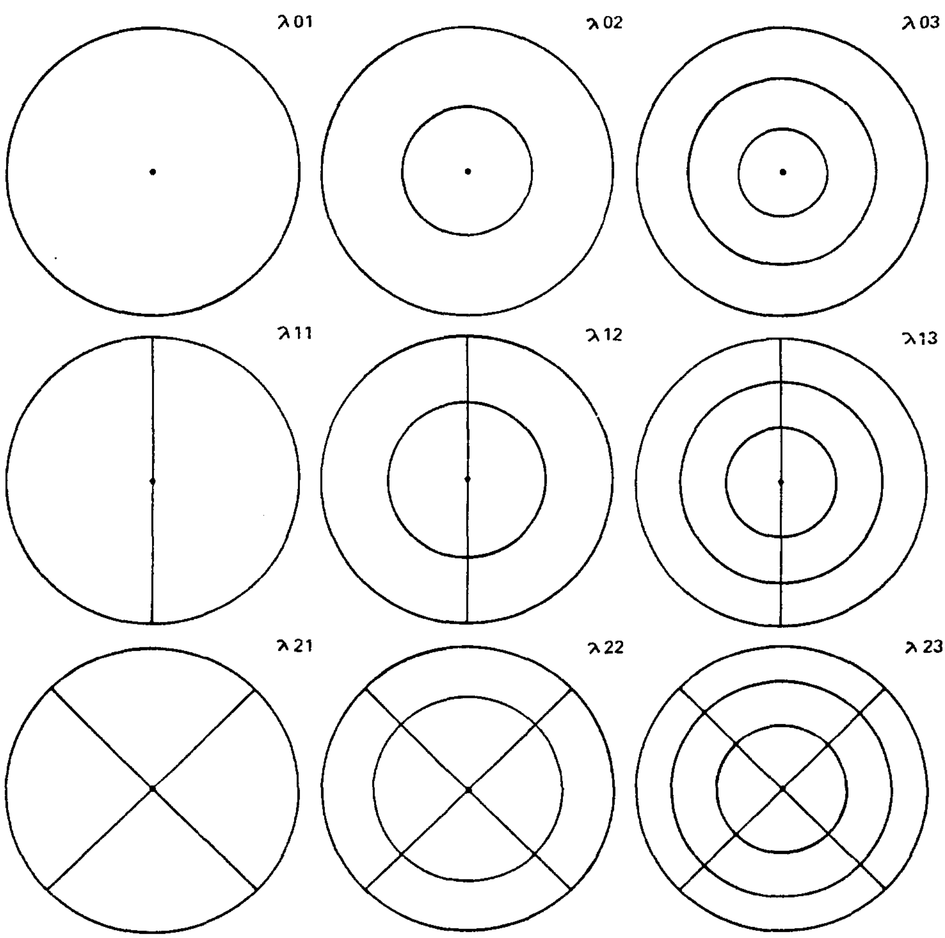

Fig. 3. Nodal lines in polar coordinates.

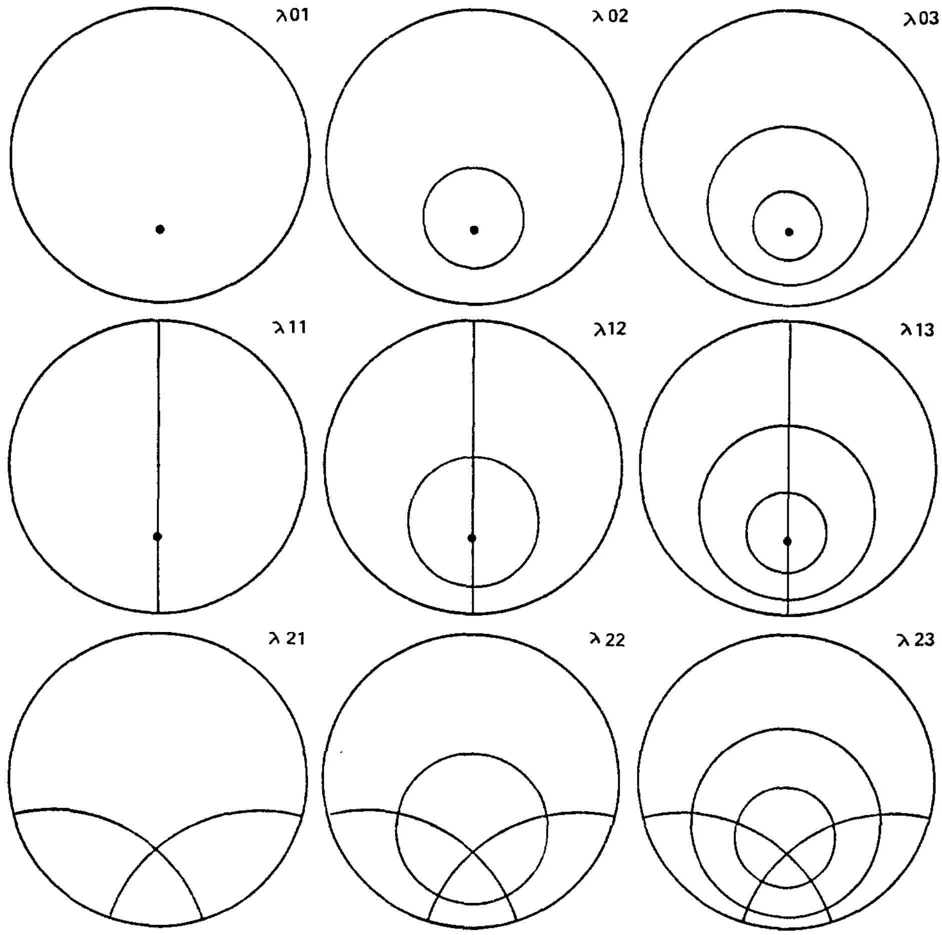

Fig. 4. Nodal lines in bipolar coordinates.

from the center to the edge of the drum, i.e., $\zeta = \frac{1}{2}$.

Since the eigenvalues of the Bessel functions are not integral multiples of each other, the solution as a function of time is not periodic and hence the tone is not what would normally be called musical. On looking at the nonsymmetrical problem it is apparent that the coaxial circles corresponding to the circular nodal paths are closer together than in the symmetrical case. From this, one can conclude that the "pitch" of the drum can be raised by striking at a point closer to the edge than the center. This observation is also an experimental fact.

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